On the supergravity description of boost invariant conformal plasma at strong coupling

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Abstract

We study string theory duals of the expanding boost invariant conformal gauge theory plasmas at strong coupling. The dual supergravity background is constructed as an asymptotic late-time expansion, corresponding to equilibration of the gauge theory plasma. The absence of curvature singularities in the first few orders of the late-time expansion of the dual gravitational background unambiguously determines the equilibrium equation of the state, and the shear viscosity of the gauge theory plasma. While the absence of the leading pole singularities in the gravitational curvature invariants at the third order in late-time expansion determines the relaxation time of the plasma, the subleading logarithmic singularity can not be canceled within a supergravity approximation. Thus, a supergravity approximation to a dual description of the strongly coupled boost invariant expanding plasma is inconsistent. Nevertheless we find that the relaxation time determined from cancellation of pole singularities is quite robust.

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1 Introduction

Gauge/string correspondence [1, 2] developed into a valuable tool to study strong coupling dynamics of gauge theory plasma, and might be relevant in understanding the physics of quark-gluon plasma (QGP) produced at RHIC in heavy ion collisions [3, 4, 5, 6]. Besides applications devoted to determine the equation of state of the gauge theory plasma at strong coupling [7, 8, 9, 10, 11], the dual string theory models have been successful in computing plasma transport properties [12, 13, 14, 15, 16, 17, 18], the quenching of partonic jets [19, 20, 21], and photon and dilepton production [22]. Much less studied are the truly dynamical processes in plasma[1] .

In [25] a framework of constructing the string theory duals to expanding boost invariant conformal plasma was proposed[2]. This was later used to study viscous properties of the expanding plasma [26, 27, 28], other applications were further discussed in [30, 31, 32, 34, 33, 35, 36, 37]. The basic idea of the approach is to set up the asymptotic geometry of the gravitational dual to that of the boost invariant frame, suggested by Bjorken [38] as a description of the central rapidity region in highly relativistic nucleus-nucleus collisions. One further has to specify the normalizable modes (a string theory duals to expectation values of appropriate gauge invariant operators in plasma) for the metric and other supergravity/string theory fields so that the singularity-free description of string theory is guaranteed. As explained in [25, 27, 28], requiring the absence of certain curvature singularities in dual gravitational backgrounds constrains the equilibrium equation of state of supersymmetric $\mathcal{N} = 4$ Yang-Mills (SYM) plasma (in agreement with [7]), its shear viscosity (in agreement with [39]), and the relaxation time of $\mathcal{N} = 4$ SYM plasma (in agreement with [40]). Despite an apparent success of the framework, it suffers a serious drawback: not all the singularities in the dual string theory description of the strongly coupled expanding $\mathcal{N} = 4$ SYM plasma have been canceled [28]. The subject of this paper is to comment on the gravitational singularities observed in [28].

To begin with, we would like to emphasize the importance of singularities in string theory. As we already mentioned, constructing a string theory dual to expanding boost-invariant plasma implied identification of a set of string theory fields with non-vanishing normalizable modes, dual to vacuum expectation values (VEVs) of the gauge

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[1]One notable example is the analysis of the shock waves generated by a heavy quark moving in gauge theory plasma [23, 24].

[2]Earlier work on more qualitative aspects of expanding plasma includes [29].

invariant operators in plasma. A priori, it is difficult to determine which operators in plasma will develop a vacuum expectation value [3]. In practice, on the dual gravitational side of the correspondence one usually truncates the full string theory to low-energy type IIB supergravity, and often further to a consistent Kaluza-Klein truncation of type IIB supergravity, keeping supergravity modes invariant under the symmetries of the problem. From the gauge theory perspective, such an approximation implies that only certain operators are assumed to develop a VEV. Typically these operators are gauge-invariant operators of low dimension. Of course, the latter assumption might, or might not be correct: luckily, on the gravitational side the consistency of the approximation is severely constrained by requiring an absence of singularities in the bulk of the gravitational background. A typical example of this phenomena is thermodynamics of cascading gauge theory [42]: if one assumes that the only operators that develop a VEV at high temperatures in this gauge theory are the same ones as those at zero temperature, all dual gravitational backgrounds will have a null singularity [43, 44]; on the other hand, turning on a VEV of an irrelevant dimension-6 operator [45] leads to a smooth geometry [46]. Thus, the presence of bulk curvature singularities in gravitational backgrounds of string theory points to an inconsistent truncation.

In [25, 27, 28] the authors assumed a truncation of the full string theory dual to $\mathcal{N} = 4$ expanding boost-invariant SYM plasma to a Kaluza-Klein reduction of type IIB supergravity on S^5, maintaining only the normalizable modes of the five-dimensional metric (dual to a stress-energy tensor of the plasma), and the dilaton normalizable mode (dual to a dimension-4 operator $\text{Tr} \, F^2$). The absence of singularities in five-dimensional Riemann tensor invariant $R^{(5)}_{\mu\nu\lambda\rho} R^{(5)\mu\nu\lambda\rho}$ of the asymptotic late-time expansion of the background geometry at leading and first two subleading orders correctly reproduced the equilibrium equation of state of $\mathcal{N} = 4$ SYM plasma and its shear viscosity. Furthermore, removing the pole singularities in $R^{(5)}_{\mu\nu\lambda\rho} R^{(5)\mu\nu\lambda\rho}$ at third order in late-time expansion determined for the first time the relaxation time of the strongly coupled $\mathcal{N} = 4$ SYM plasma [28]. Given the truncation of the string theory, it was not possible to cancel a logarithmic singularity at third order in $R^{(5)}_{\mu\nu\lambda\rho} R^{(5)\mu\nu\lambda\rho}$. Turning on an appropriate dilaton mode removes a logarithmic singularity in the five-dimensional string frame metric in $R^{(5,string)}_{\mu\nu\lambda\rho} R^{(5,string)\mu\nu\lambda\rho}$. However, as we show in the present paper, the full 10 dimensional metric remains singular even in the string frame.

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[3]For a weak coupling analysis in QGP pointing to a development of a nonzero expectation value for $\text{Tr} \, F^2$ see [41].

The paper is organized as follows. In the next section we review the boost invariant kinematics of gauge theory plasma. We identify parameters in the late-time expansion of the stress-energy tensor one-point correlation function with the hydrodynamic parameters of Müller-Israel-Stewart transient theory [47, 48]. Following the idea that a presence of singularities indicates an inconsistent truncation, we consider in the section 3 the full $SO(6)$ invariant sector of type IIB supergravity, assuming the parity invariance of the $\mathcal{N} = 4$ SYM expanding plasma. Compare to the analysis in [28], this includes one additional $SO(6)$ invariant scalar, dual to a dimension-8 operator. We show that using this additional scalar one can cancel the logarithmic singularity at the third order in late-time expansion either in ten-dimensional Einstein frame Ricci tensor squared $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, or in ten-dimensional Einstein frame Riemann tensor squared $\mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho}$, but not *both* the singularities. Given the *assumed* symmetries of the expanding plasma system, there are no additional supergravity fields to be turned on to cancel the singularity in string theory dual to expanding $\mathcal{N} = 4$ SYM plasma. In section 4 we further show that canceling a singularity typically requires additional supergravity (or string theory) fields to be turned on. Using the consistent truncation presented in [45, 50], we consider supergravity dual to superconformal Klebanov-Witten gauge theory [51] plasma. Here, in addition to a dimension-8 operator (analogous to the one discussed in section 3) there is an additional scalar, dual to a dimension-6 operator [45]. We show that turning on the latter scalar can remove logarithmic singularities in the third order in the late-time expansion of the ten-dimensional Einstein frame Ricci scalar $\mathcal{R}$, as well as $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho}$. However, we find new logarithmic singularities at the third order in higher curvature invariants such as $\mathcal{R}_{\mu_1\nu_1\lambda_1\rho_1}\mathcal{R}^{\mu_1\nu_1\lambda_2\rho_2}\mathcal{R}_{\mu_2\nu_2}{}^{\lambda_1\rho_1}\mathcal{R}^{\mu_2\nu_2}{}_{\lambda_2\rho_2}$, as well as logarithmic singularities with different coefficients in $(\mathcal{R}_{....})^8$, $(\mathcal{R}_{....})^{16}$ and so on. Canceling the logarithmic singularities in all the curvature invariants appears to require an infinite number of fields. Since such fields are absent in supergravity approximation to string theory dual of expanding boost invariant plasma, we conclude that such a supergravity approximation is inconsistent. In section 5 we present further directions in analyzing the time-dependent framework proposed in [25, 27, 28].

2   Review of boost-invariant kinematics

Quark-gluon plasma is expected to be produced in high-energy collisions of heavy ions. In nucleus-nucleus experiments a central rapidity plateau structure for particle formation has been observed. Specifically, the expansion of the plasma appears to be longitudinal and homogeneous near the collision axis in the case of central collisions. Thus the system is expected to be boost-invariant in the the longitudinal plane. It is convenient to introduce the proper-time τ and the rapidity y, defined as

$$x^0 = \tau \cosh y \,, \qquad x^3 = \tau \sinh y \,. \qquad (2.1)$$

Then rapidity invariance amounts to independence on y. We will also assume $y \to -y$ invariance.

We will also make another approximation, following [38], namely independence on the transverse coordinates $x_\perp$. This corresponds to the limit of very large nuclei.

With these assumptions, the whole energy-momentum tensor can be expressed purely in terms of a single function[4] $\epsilon(\tau)$, the energy density in the local rest frame. This remaining function should be fixed by gauge theory dynamics.

The proposal of [25] to use the AdS/CFT correspondence to determine $\epsilon(\tau)$ amounts to constructing a dual geometry, by solving supergravity equations with the boundary conditions

$$\langle T_{\mu\nu} \rangle = \frac{N_c^2}{2\pi} \lim_{z \to 0} \frac{g_{\mu\nu} - \eta_{\mu\nu}}{z^4} \,, \qquad (2.2)$$

where the (5-dimensional part of the) metric is written in the Fefferman-Graham form:

$$ds_{5dim.}^2 = \frac{g_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2} \,. \qquad (2.3)$$

Given any $\epsilon(\tau)$ it is possible to construct such a metric. In [25] it was advocated that the requirement of the non-singularity of this geometry picks out a physically acceptable $\epsilon(\tau)$. In [25, 27, 28] this problem was analyzed in an expansion for large τ. At each order the square of the Riemann curvature tensor had poles which could be canceled only by a specific choice of the coefficients of $\epsilon(\tau)$. In the third coefficient there remained a leftover logarithmic singularity in the Einstein frame[5] . In this paper we would like to revisit this issue.

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[4]After taking into account energy-momentum conservation and tracelessness. For explicit formulas see [25].

[5]The is also a singularity in the string frame Ricci tensor squared $R_{\mu\nu}^{(10,string)} R^{(10,string)\mu\nu}$.

The outcome of this calculation is a specific τ dependence for $\epsilon(\tau)$:

$$\epsilon(\tau) = \frac{1}{\tau^{\frac{4}{3}}} \left(1 - \frac{\cdots}{\tau^{\frac{2}{3}}} + \frac{\cdots}{\tau^{\frac{4}{3}}} + \ldots \right) . \tag{2.4}$$

Now we may obtain the physical interpretation of the above result by checking whether it is a solution of some specific phenomenological equations. In the above case, the leading behavior corresponds to perfect fluid hydrodynamics, the subleading one is related to the effect of shear viscosity, while the third one is related to second order viscous hydrodynamics relaxation time. Let us note that the relaxation time depends on the form of phenomenological equations used to interpret (2.4).

Müller-Israel-Stewart theory [47, 48] of dissipative processes provides a nice framework to study relativistic fluid dynamics. In the Bjorken regime of boost invariant expansion of plasma, the transport equations take the form [49]

$$\begin{aligned}
0 =& \frac{d\epsilon}{d\tau} + \frac{\epsilon + p}{\tau} - \frac{1}{\tau}\Phi \\
0 =& \frac{d\Phi}{d\tau} + \frac{\Phi}{\tau_\pi} + \frac{1}{2}\Phi \left(\frac{1}{\tau} + \frac{1}{\beta_2}T\frac{d}{d\tau}\left(\frac{\beta_2}{T}\right) \right) - \frac{2}{3}\frac{1}{\beta_2}\frac{1}{\tau} ,
\end{aligned} \tag{2.5}$$

where τ_π is the relaxation time, ϵ is the energy density, p is the pressure, Φ is related to the dissipative part of the energy-momentum, and

$$\beta_2 = \frac{\tau_\pi}{2\eta} , \tag{2.6}$$

with η being the shear viscosity. In (2.5) anticipating application to conformal gauge theory plasma we set the bulk viscosity ζ to zero.

In the case of the $\mathcal{N} = 4$ SYM plasma we have

$$\epsilon(\tau) = \frac{3}{8}\pi^2 N^2 \, T(\tau)^4 , \qquad p(\tau) = \frac{1}{3}\epsilon(\tau) , \qquad \eta(\tau) = A \, s(\tau) = A \, \frac{1}{2}\pi^2 N^2 \, T(\tau)^3 ,$$

$$\tau_\pi(\tau) = r \, \tau_\pi^{\text{Boltzmann}}(\tau) = r \, \frac{3\eta(\tau)}{2p(\tau)} .$$

$$\tag{2.7}$$

Given (2.7), we can solve (2.5) perturbatively as $\tau \to \infty$:

$$T(\tau) = \frac{\Lambda}{\tau^{1/3}} \left(1 + \sum_{k=1}^{\infty} \frac{t_k}{(\Lambda\tau^{2/3})^k} \right) , \qquad \Phi(\tau) = \frac{2}{3}\pi^2 N^2 \, A \, \frac{\Lambda^3}{\tau^2} \left(1 + \sum_{k=1}^{\infty} \frac{f_k}{(\Lambda\tau^{2/3})^k} \right) ,$$

$$\tag{2.8}$$

where Λ is an arbitrary scale. The constant coefficients $\{t_k, f_k\}$ are related to the $\mathcal{N} = 4$ SYM dimensionless transport parameters A (the ratio of shear-to-entropy density) and r (the plasma relaxation time in units of Boltzmann relaxation time). For the first few coefficients we find:

$$t_1 = -\frac{2}{3}A\,, \qquad t_2 = -\frac{4}{3}A^2 r\,, \qquad t_3 = -\frac{8}{27}A^3 r(1 + 24r)\,,$$

$$f_1 = 2A(2r - 1)\,, \qquad f_2 = \frac{4}{3}A^2(1 - 8r + 24r^2)\,, \tag{2.9}$$

$$f_3 = \frac{8}{27}A^3(-1 + 24r - 234r^2 + 1296r^3)\,.$$

From the supergravity computations first done in [28] (and reproduced in the following section of this paper) we find

$$\epsilon(\tau) = \left(\frac{N^2}{2\pi^2}\right) \frac{1}{\tau^{4/3}}\left\{1 - \frac{2\eta_0}{\tau^{2/3}} + \left(\frac{10}{3}\eta_0^2 + \frac{C}{36}\right)\frac{1}{\tau^{4/3}} + \cdots\right\}. \tag{2.10}$$

Matching the gauge theory expansion for the energy density (2.7) with that of the dual gravitational description (2.10) we find

$$\Lambda = \frac{\sqrt{2}}{3^{1/4}\pi}\,, \qquad A = \frac{3^{3/4}}{2^{3/2}\pi}\,\eta_0\,, \qquad r = -\frac{11}{18} - \frac{1}{108}\frac{C}{\eta_0^2}\,. \tag{2.11}$$

Using the supergravity results (3.29) and (3.33), we find

$$A = \frac{1}{4\pi}\,, \qquad r = \frac{1}{3}(1 - \ln 2)\,. \tag{2.12}$$

A different formulation of second order hydrodynamics in [52] derived from Boltzmann equations lead to the equation

$$0 = \frac{d\epsilon}{d\tau} + \frac{\epsilon + p}{\tau} - \frac{1}{\tau}\Phi$$
$$0 = \frac{d\Phi}{d\tau} + \frac{\Phi}{\tau_\pi^{(ii)}} - \frac{2}{3}\frac{1}{\beta_2}\frac{1}{\tau}\,, \tag{2.13}$$

which provides a different definition of the relaxation time $\tau_\pi^{(ii)}$. Using (2.13) leads to $r^{(ii)} = \frac{1}{9}(1 - \ln 2)$ quoted in [28]. Of course (2.5) and (2.13) provide different interpretations of the same $\epsilon(\tau)$. Without lifting the symmetry assumptions of the uniform boost invariant plasma expansion we may not rule out one of the (2.5) and (2.13) descriptions.

3 $\mathcal{N} = 4$ QGP

Supergravity dual to boost invariant expanding $\mathcal{N} = 4$ QGP was discussed in [25, 27, 28]. We extend the previous analysis including *all* supergravity modes, invariant under the symmetries of the problem. Specifically, we assume that the $SO(6)$ R-symmetry of $\mathcal{N} = 4$ SYM is unbroken. We further assume that the boost invariant plasma is invariant under separate reflections of all three spatial directions. The latter assumption in particular ensures that the supergravity dual excludes the axion and various 3-form fluxes along extended spacial directions of the boundary — say RR fluxes with components $F_{\tau x_1 x_2}$ within the metric ansatz (3.11). In section 5 we comment why we believe that even allowing for a parity violating supergravity modes would not change our main conclusion, *i.e.*, the inconsistency of the supergravity approximation in dual description of boost invariants expanding plasmas.

Given the symmetry assumptions of the problem as well as the approximation of the full string theory by its low energy type IIB supergravity, the non-vanishing fields we need to keep is the five-dimensional metric, the warp factor of the five-sphere, the self-dual five-form flux and the dilaton. Parity invariance and the symmetries of the boost invariant frame further restricts the five-dimensional metric as discussed below.

3.1 Consistent Kaluza-Klein Reduction

Consider Einstein frame type IIB low-energy effective action in 10-dimensions

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} d^{10}\xi \sqrt{-\tilde{g}} \left\{ \mathcal{R} - \frac{1}{2} \left(\tilde{\partial}\phi \right)^2 - \frac{1}{4 \cdot 5!} F_5^2 \right\}, \qquad (3.1)$$

with metric ansatz:

$$\begin{aligned} d\tilde{s}_{10}^2 &= \tilde{g}_{MN} d\xi^M d\xi^N = \\ &= \sigma^{-2}(x) g_{\mu\nu}(x) dx^\mu dx^\nu + \sigma^{6/5}(x) \left(dS^5 \right)^2, \end{aligned} \qquad (3.2)$$

where $M, N, \ldots = 0, \ldots, 9$ and $\mu, \nu, \ldots = 0, \ldots, 4$, and $(dS^5)^2$ is the line element for a 5-dimensional sphere with unit radius. For the 5-form F_5 we assume that

$$F_5 = \mathcal{F}_5 + \star \mathcal{F}_5, \qquad \mathcal{F}_5 = -4Q \, \omega_{S^5}, \qquad (3.3)$$

where ω_{S^5} is the 5-sphere volume form and Q is a constant. We further assume that the dilaton is $\phi = \phi(x)$.

With the ansatz (3.2), we find

$$
\begin{aligned}
\sqrt{-\tilde{g}} &= \sigma^{-2}\sqrt{-g}\sqrt{g_{(S^5)}} \\
\mathcal{R} &= \sigma^2 \left\{ R - \frac{24}{5}(\partial \ln \sigma)^2 \right\} + 20\sigma^{-6/5} \\
-\frac{1}{4 \cdot 5!}F_5^2 &= -8Q^2\sigma^{-6} \\
-\frac{1}{2}\left(\tilde{\partial}\phi\right)^2 &= -\frac{1}{2}(\partial\phi)^2 \; \sigma^2 \,,
\end{aligned}
\tag{3.4}
$$

where R is the Ricci scalar for the 5-dimensional metric $g_{\mu\nu}$. The 5-dimensional effective action therefore takes the following form:

$$
S_5^{\mathrm{eff}} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}_5} d^5x \sqrt{-g} \left\{ R - \frac{1}{2}(\partial\phi)^2 - \frac{24}{5}(\partial \ln \sigma)^2 - \mathcal{P}(\sigma) \right\},
\tag{3.5}
$$

with

$$
\mathcal{P}(\sigma) = -20\sigma^{-16/5} + 8Q^2\sigma^{-8}, \qquad \kappa_5^2 = \frac{\kappa_{10}^2}{\mathrm{vol}\left\{S^5\right\}}.
\tag{3.6}
$$

It is convenient to introduce a scalar field $\alpha(x)$ defined as

$$
\sigma(x) = e^{\alpha(x)},
\tag{3.7}
$$

so that (3.5) can be rewritten as:

$$
S_5^{\mathrm{eff}} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}_5} d^5x \sqrt{-g} \left\{ R - \frac{1}{2}(\partial\phi)^2 - \frac{24}{5}(\partial\alpha)^2 - \mathcal{P}(\alpha) \right\},
\tag{3.8}
$$

where

$$
\mathcal{P}(\alpha) = -20\,e^{-16\alpha/5} + 8Q^2\,e^{-8\alpha}.
\tag{3.9}
$$

From (3.8), the Einstein equations and the equation of motion for α are respectively given by:

$$
\begin{aligned}
R_{\mu\nu} &= \frac{24}{5}(\partial_\mu\alpha)(\partial_\nu\alpha) + \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) + \frac{1}{3}g_{\mu\nu}\mathcal{P}(\alpha) \\
\Box\alpha &= \frac{5}{48}\frac{\partial\mathcal{P}}{\partial\alpha} \\
\Box\phi &= 0\,.
\end{aligned}
\tag{3.10}
$$

9

3.2 Equations of motion

For the five-dimensional metric we use the same ansatz as in [25, 27, 28]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu =$$
$$= \frac{1}{z^2}\left[-e^{2a(\tau,z)}d\tau^2 + e^{2b(\tau,z)}\tau^2 dy^2 + e^{2c(\tau,z)}dx_\perp^2\right] + \frac{dz^2}{z^2}, \tag{3.11}$$

where $dx_\perp^2 \equiv dx_1^2 + dx_2^2$. It is easy to see that this is the most general boost invariant geometry written in Fefferman-Graham coordinates subject to a parity invariance along the boost direction[6].

Further assuming $\alpha = \alpha(\tau, z)$ and $\phi = \phi(\tau, z)$, the equations of motion for the background (3.11), the dilaton ϕ and the scalar α become:

- Einstein equations

$$e^{2a}\left[\partial_z^2 a + (\partial_z a)^2 + (\partial_z a)(\partial_z b) + 2(\partial_z a)(\partial_z c) - \frac{4}{z}\partial_z a - \frac{1}{z}\partial_z b - \frac{2}{z}\partial_z c + \frac{4}{z^2}\right]$$
$$- \left[\partial_\tau^2 b + (\partial_\tau b)^2 - (\partial_\tau a)(\partial_\tau b) + 2\left(\partial_\tau^2 c - (\partial_\tau c)^2 - (\partial_\tau a)(\partial_\tau c)\right) - \frac{1}{\tau}\partial_\tau a + \frac{2}{\tau}\partial_\tau b\right]$$
$$= \frac{24}{5}(\partial_\tau \alpha)^2 + \frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{3}\frac{e^{2a}}{z^2}\mathcal{P}(\alpha),$$

$$\tag{3.12}$$

$$e^{2a}\left[\partial_z^2 b + (\partial_z b)^2 + (\partial_z a)(\partial_z b) + 2(\partial_z b)(\partial_z c) - \frac{1}{z}\partial_z a - \frac{4}{z}\partial_z b - \frac{2}{z}\partial_z c + \frac{4}{z^2}\right]$$
$$- \left[\partial_\tau^2 b + (\partial_\tau b)^2 - (\partial_\tau a)(\partial_\tau b) + 2(\partial_\tau b)(\partial_\tau c) - \frac{1}{\tau}\partial_\tau a + \frac{2}{\tau}\partial_\tau b + \frac{2}{\tau}\partial_\tau c\right]$$
$$= -\frac{1}{3}\frac{e^{2a}}{z^2}\mathcal{P}(\alpha),$$

$$\tag{3.13}$$

$$e^{2a}\left[\partial_z^2 c + 2(\partial_z c)^2 + (\partial_z a)(\partial_z c) + (\partial_z b)(\partial_z c) - \frac{1}{z}\partial_z a - \frac{1}{z}\partial_z b - \frac{5}{z}\partial_z c + \frac{4}{z^2}\right]$$
$$- \left[\partial_\tau^2 c + 2(\partial_\tau c)^2 - (\partial_\tau a)(\partial_\tau c) + (\partial_\tau b)(\partial_\tau c) + \frac{1}{\tau}(\partial_\tau c)\right]$$
$$= -\frac{1}{3}\frac{e^{2a}}{z^2}\mathcal{P}(\alpha),$$

$$\tag{3.14}$$

---

[6]The parity invariance excludes metric components $g_{yz}(\tau, z)$ and $g_{\tau y}(\tau, z)$, in principle allowed by a Fefferman-Graham coordinate frame.

$$\partial_z^2 a + (\partial_z a)^2 + \partial_z^2 b + (\partial_z b)^2 + 2\left[\partial_z^2 c + (\partial_z c)^2\right] - \frac{1}{z}\partial_z a - \frac{1}{z}\partial_z b - \frac{2}{z}\partial_z c + \frac{4}{z^2}$$
$$= -\frac{24}{5}(\partial_z \alpha)^2 - \frac{1}{2}(\partial_z \phi)^2 - \frac{1}{3}\frac{\mathcal{P}(\alpha)}{z^2},$$

(3.15)

$$\partial_\tau \partial_z b - (\partial_\tau b)(\partial_z a) + (\partial_\tau b)(\partial_z b) + 2\left[\partial_\tau \partial_z c + (\partial_\tau c)(\partial_z c) - (\partial_\tau c)(\partial_z a)\right]$$
$$- \frac{1}{\tau}\partial_z a + \frac{1}{\tau}\partial_z b = -\frac{24}{5}(\partial_\tau \alpha)(\partial_z \alpha) - \frac{1}{2}(\partial_\tau \phi)(\partial_z \phi) \; ;$$

(3.16)

- dilaton equation:

$$0 = z^2 \left\{ \partial_z^2 \phi + (\partial_z a + \partial_z b + 2\partial_z c)(\partial_z \phi) - \frac{3}{z}(\partial_z \phi) \right.$$
$$\left. - e^{-2a}\left[\partial_\tau^2 \phi + \left(-\partial_\tau a + \partial_\tau b + 2\partial_\tau c + \frac{1}{\tau}\right)(\partial_\tau \phi)\right]\right\} \; ;$$

(3.17)

- equation of motion for α

$$\frac{5}{48}\frac{\partial \mathcal{P}}{\partial \alpha} = z^2 \left\{ \partial_z^2 \alpha + (\partial_z a + \partial_z b + 2\partial_z c)(\partial_z \alpha) - \frac{3}{z}(\partial_z \alpha) \right.$$
$$\left. - e^{-2a}\left[\partial_\tau^2 \alpha + \left(-\partial_\tau a + \partial_\tau b + 2\partial_\tau c + \frac{1}{\tau}\right)(\partial_\tau \alpha)\right]\right\} \; .$$

(3.18)

3.3   Late-time expansion

In [25, 26, 27, 28] equations similar to (3.12)-(3.17) were solved asymptotically as a late-time expansion in powers of $(\tau^{-2/3})$ (or powers of $(\tau^{-1/3})$ for the dilaton) introducing the scaling variable

$$v \equiv \frac{z}{\tau^{1/3}}$$

(3.19)

in the limit $\tau \to \infty$, with v kept fixed. Such a scaling is well motivated on physical grounds as in this case we find that the boundary energy density $\epsilon(\tau)$ extracted from the one-point correction function of the boundary stress-energy tensor

$$\epsilon(\tau) = -\frac{N^2}{2\pi^2}\lim_{z\to 0}\frac{2a(z,\tau)}{z^4} = -\frac{N^2}{2\pi^2}\lim_{v\to 0}\frac{2a(v,\tau)}{v^4\tau^{4/3}}$$

(3.20)

would have a late-τ expansion appropriate for a conformal plasma in Bjorken regime [28]. Thus we expect

$$a(\tau,v) = a_0(v) + \frac{1}{\tau^{2/3}}a_1(v) + \frac{1}{\tau^{4/3}}a_2(v) + \frac{1}{\tau^2}a_3(v) + \mathcal{O}(\tau^{-8/3})$$
$$b(\tau,v) = b_0(v) + \frac{1}{\tau^{2/3}}b_1(v) + \frac{1}{\tau^{4/3}}b_2(v) + \frac{1}{\tau^2}b_3(v) + \mathcal{O}(\tau^{-8/3})$$
$$c(\tau,v) = c_0(v) + \frac{1}{\tau^{2/3}}c_1(v) + \frac{1}{\tau^{4/3}}c_2(v) + \frac{1}{\tau^2}c_3(v) + \mathcal{O}(\tau^{-8/3}) \; .$$

(3.21)

In the equilibrium, both the dilaton and the α-scalar vanish, implying that in (3.12)-(3.16) they enter quadratically for small ϕ, α. Thus, given (3.21) we have two possible late-time asymptotic expansions for ϕ and α:

$$\phi(\tau, v) = \frac{1}{\tau^{2/3}}\phi_1(v) + \frac{1}{\tau^{4/3}}\phi_2(v) + \frac{1}{\tau^2}\phi_3(v) + \mathcal{O}(\tau^{-8/3})$$

or   (3.22)

$$\phi(\tau, v) = \frac{1}{\tau^{1/3}}\hat{\phi}_1(v) + \frac{1}{\tau}\hat{\phi}_2(v) + \frac{1}{\tau^{5/3}}\hat{\phi}_3(v) + \mathcal{O}(\tau^{-7/3}),$$

and

$$\alpha(\tau, v) = \frac{1}{\tau^{2/3}}\alpha_1(v) + \frac{1}{\tau^{4/3}}\alpha_2(v) + \frac{1}{\tau^2}\alpha_3(v) + \mathcal{O}(\tau^{-8/3})$$

or   (3.23)

$$a(\tau, v) = \frac{1}{\tau^{1/3}}\hat{\alpha}_1(v) + \frac{1}{\tau}\hat{\alpha}_2(v) + \frac{1}{\tau^{5/3}}\hat{\alpha}_3(v) + \mathcal{O}(\tau^{-7/3}).$$

For each of the four possible asymptotic expansions, we solve (3.12)-(3.18) subject to the boundary conditions

$$\left. \left\{ a_i(v), b_i(v), c_i(v) \right\} \right|_{v \to 0} = 0, \qquad \left. \left\{ \phi_i(v) \text{ or } \hat{\phi}_i(v) \right\} \right|_{v \to 0} = 0,$$

$$\left. \left\{ \alpha_i(v) \text{ or } \hat{\alpha}_i(v) \right\} \right|_{v \to 0} = 0.$$

(3.24)

Additionally, we will require the absence of singularities in the asymptotic late-time expansion of the quadratic curvature invariant $\mathcal{I}^{[2]}$:

$$\mathcal{I}^{[2]} \equiv \mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}$$

$$= \mathcal{I}_0^{[2]}(v) + \frac{1}{\tau^{2/3}}\mathcal{I}_1^{[2]}(v) + \frac{1}{\tau^{4/3}}\mathcal{I}_2^{[2]}(v) + \frac{1}{\tau^2}\mathcal{I}_3^{[2]}(v) + \mathcal{O}(\tau^{-8/3}).$$

(3.25)

Of course, ultimately, all the curvature invariants of the bulk geometry must be nonsingular for the supergravity approximation dual to boost invariant $\mathcal{N} = 4$ SYM plasma to be consistent, unless singularities are hidden behind the horizon. However, it turns out that asymptotic expansions (3.21)-(3.23) are fixed unambiguously, given (3.24) and (3.25). Thus non-singularity of the other curvature invariants provides a strong consistency check on the validity of the supergravity approximation.

3.4   Solution of the late-time series and curvature singularities

The absence of curvature singularities in $\mathcal{I}_0^{[2]}$ determines the leading solution to be [25]

$$a_0 = \frac{1}{2}\ln\frac{(1 - v^4/3)^2}{1 + v^4/3}, \qquad b_0 = c_0 = \frac{1}{2}\ln\left(1 + v^4/3\right).$$

(3.26)

Notice that $v = 3^{1/4}$ is the horizon to leading order in τ. It is straightforward to find that the absence of naked singularities in the bulk (specifically at $v = 3^{1/4}$) in $\mathcal{I}^{[2]}_{1,2}$ would require that $\alpha_1 = \alpha_2 = 0$ $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$, $\phi_1 = \phi_2$ and $\hat{\phi}_1 = \hat{\phi}_2 = 0$. We further have [26, 27, 28]

$$
a_1 = \eta_0 \frac{(9 - v^4)\, v^4}{9 - v^8}\,, \qquad c_1 = -\eta_0 \frac{v^4}{3 + v^4} - \frac{\eta_0}{2}\, \ln \frac{3 - v^4}{3 + v^4}\,,
$$
$$
b_1 = -3\eta_0 \frac{v^4}{3 + v^4} - 2c_1\,,
$$

(3.27)

and

$$
\begin{aligned}
a_2 =& \frac{(9 + 5v^4)v^2}{12(9 - v^8)} - C\frac{(9 + v^4)v^4}{72(9 - v^8)} + \eta_0^2 \frac{(-1053 - 171v^4 + 9v^8 + 7v^{12})v^4}{6(9 - v^8)^2} \\
&+ \frac{1}{8\sqrt{3}} \ln \frac{\sqrt{3} - v^2}{\sqrt{3} + v^2} - \frac{3}{4}\eta_0^2 \ln \frac{3 - v^4}{3 + v^4} \\
c_2 =& -\frac{\pi^2}{288\sqrt{3}} + \frac{v^2(9 + v^4)}{12(9 - v^8)} + C\frac{v^4}{72(3 + v^4)} - \eta_0^2 \frac{(-9 + 54v^4 + 7v^8)v^4}{6(3 + v^4)(9 - v^8)} \\
&+ \frac{1}{8\sqrt{3}} \ln \frac{\sqrt{3} - v^2}{\sqrt{3} + v^2} + \frac{1}{72}(C + 66\eta_0^2) \ln \frac{3 - v^4}{3 + v^4} \\
&+ \frac{1}{24\sqrt{3}} \left(\ln \frac{\sqrt{3} - v^2}{\sqrt{3} + v^2} \ln \frac{(\sqrt{3} - v^2)(\sqrt{3} + v^2)^3}{4(3 + v^4)^2} - \mathrm{li}_2 \left(-\frac{(\sqrt{3} - v^2)^2}{(\sqrt{3} + v^2)^2} \right) \right) \\
b_2 =& -2c_2 + \frac{v^2}{4(3 + v^4)} + C\frac{v^4}{24(3 + v^4)} + \eta_0^2 \frac{(39 + 7v^4)v^4}{2(3 + v^4)^2} + \frac{1}{8\sqrt{3}} \ln \frac{\sqrt{3} - v^2}{\sqrt{3} + v^2} \\
&+ \frac{3}{4}\eta_0^2 \ln \frac{3 - v^4}{3 + v^4}\,,
\end{aligned}
$$

(3.28)

with [27]

$$
\eta_0 = \frac{1}{2^{1/2}3^{3/4}}\,.
$$

(3.29)

While the equations at the third order for $\{a_3, b_3, c_3\}$ are a bit complicated, it is possible to decouple the equation for a_3. The resulting equation is too long to be presented here. It is straightforward to solve the equation perturbatively as

$$
y \equiv 3 - v^4 \to 0_+\,,
$$

(3.30)

$$
\frac{da_3}{dy} = -\frac{\sqrt{6}}{288}\, y^{-4} + \left(\frac{2^{1/2}3^{1/4}}{48} - \frac{2^{1/2}3^{1/4} \ln 2}{144} \right)\, y^{-3} + \mathcal{O}\left(y^{-2}\right)\,,
$$

(3.31)

which is sufficient to determine the singularities in $\mathcal{I}^{[2]}_3$ at $v = 3^{1/4}$.

13

The absence of pole singularities in $\mathcal{I}_3^{[2]}$ at $v = 3^{1/4}$ implies that only $\alpha_3 \neq 0$:

$$\alpha_3 = \alpha_{3,0} \left(\left(\frac{1}{96v^4} + \frac{v^4}{864} \right) \ln \frac{3 + v^4}{3 - v^4} - \frac{1}{144} \right), \tag{3.32}$$

where $\alpha_{3,0}$ is an arbitrary constant; it further constraints [28]

$$C = 2\sqrt{3} \ln 2 - \frac{17}{\sqrt{3}}. \tag{3.33}$$

Using (3.31)-(3.33) we find

$$\mathcal{I}_3^{[2]} = \text{finite} + \left(8 \; 2^{1/2} \; 3^{3/4} + \frac{14}{3} \alpha_{3,0} \right) \ln(3 - v^4), \qquad v \to 3_-^{1/4}. \tag{3.34}$$

Notice that with $\alpha_{3,0} = 0$ the logarithmic singularity agrees with the one found in [28]. From (3.34) it appears that the curvature singularities in the bulk of the supergravity dual to expanding $\mathcal{N} = 4$ SYM plasma can be canceled for an appropriate choice of $\alpha_{3,0}$. While the Ricci scalar $\mathcal{R}$ is indeed nonsingular, unfortunately, this is not the case for the square of the Ricci tensor. We find

$$\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} = \text{finite} + \frac{1}{\tau^2} \frac{40}{3} \alpha_{3,0} \ln(3 - v^4), \qquad v \to 3_-^{1/4}. \tag{3.35}$$

Thus, it is impossible to cancel logarithmic singularity both in $\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\lambda} \mathcal{R}^{\mu\nu\rho\lambda}$ at order $\mathcal{O}(\tau^{-2})$ with a supergravity field α.

3.5 Curvature singularities of the string frame metric

In [28] it was suggested that the logarithmic singularity in the curvature invariants of the Einstein frame metric (3.2) might be canceled in string frame metric

$$g_{\mu\nu}^{string} \equiv e^{\phi/2} \; g_{\mu\nu}. \tag{3.36}$$

Unfortunately, this is not possible. Indeed, as in [28], in order to avoid pole singularities in the curvature invariants of the string frame metric at leading and first three subleading orders, the dilaton can contribute only at order $\mathcal{O}(\tau^{-2})$

$$\phi(\tau, v) = \frac{1}{\tau^2} \; k_3 \; \ln \frac{3 - v^4}{3 + v^4} + \mathcal{O}\left(\tau^{-8/3}\right), \tag{3.37}$$

where k_3 is an arbitrary constant.

We find then that

$$R_{\mu\nu}^{(10,string)} R^{(10,string)\mu\nu} = \text{finite} + \frac{1}{\tau^2} \left(\frac{40}{3} \alpha_{3,0} - 160 \; k_3 \right) \ln(3 - v^4), \tag{3.38}$$

and[7]

$$R^{(10,string)}_{\mu\nu\rho\lambda} R^{(10,string)\mu\nu\rho\lambda} = \text{finite} + \frac{1}{\tau^2} \left(8 \ 2^{1/2} \ 3^{3/4} + \frac{14}{3} \ \alpha_{3,0} - 152 \ k_3 \right) \ \ln(3 - v^4),$$
(3.39)

as $v \to 3^{1/4}_-$. While it is possible to remove logarithmic singularities in (3.38) and (3.39) by properly adjusting $\alpha_{3,0}$ and k_3, the higher curvature invariants of the string frame metric would remain necessarily singular. Specifically, for string frame fourth order curvature invariants $\mathcal{I}^{(10,string)[4]}_3$ we find

$$\begin{aligned}
\mathcal{I}^{(10,string)[4]}_3 &\equiv R^{(10,string)\mu_1\nu_1}_{\mu_1\nu_1\mu\nu} R^{(10,string)\mu_1\nu_1}{}_{\rho\lambda} \ R^{(10,string)}_{\mu_2\nu_2}{}^{\mu\nu} R^{(10,string)\mu_2\nu_2\rho\lambda} \\
&= \text{finite} + \frac{1}{\tau^2} \left(\frac{640}{3} \ 2^{1/2} \ 3^{3/4} + \frac{400}{3} \ \alpha_{3,0} - 3904 \ k_3 \right) \ \ln(3 - v^4),
\end{aligned}$$
(3.40)

as $v \to 3^{1/4}_-$.

4   KW QGP

In the previous section we studied supergravity dual to $\mathcal{N} = 4$ SYM plasma assuming unbroken $SO(6)$ global symmetry and the parity invariance along the extended boundary spacial directions. At a technical level, we found that the absence of bulk singularities in the gravitational dual of expanding boost invariant plasma is linked to nontrivial profiles of massive supergravity modes (corresponding to VEV's of irrelevant gauge invariant operators in plasma). This suggests that turning on additional massive supergravity modes might remove curvature singularities. Within the assumed unbroken symmetries of the $\mathcal{N} = 4$ plasma there are no additional massive modes in the supergravity approximation. However, we can test the link between massive supergravity modes and singularities in the supergravity dual of expanding boost invariant conformal plasmas for a slightly more complicated example.

In this section we study supergravity dual to boost invariant expanding Klebanov-Witten superconformal plasma [51]. Assuming that parity along extended boundary spacial directions as well as the global $SU(2) \times SU(2) \times U(1)$ symmetry of KW plasma at equilibrium is unbroken, the effective supergravity description will contain two massive modes [45, 50] — a supergravity mode dual to a dimension-8 operator (an analog of α field in the context of $\mathcal{N} = 4$ plasma) and a supergravity mode dual to a dimension-6 operator. We expect that appropriately exciting both modes we can remove curvature

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[7]The result for the Riemann tensor squared here corrects the expression presented in [28].

singularities in all quadratic invariants of the metric curvature at the third order in late-time expansion. While we show that the latter expectation is correct, we also find that higher curvature invariants will remain singular in this model. In fact, it appears we need an infinite set of massive fields (which is not possible in the supergravity approximation) to have a nonsingular metric.

Since the computations for the most part mimic the analysis of the previous section, we highlight only the main results.

4.1 Consistent Kaluza-Klein reduction

Consistent KK reduction of the KW gauge theory plasma has been constructed in [45, 50].

The five dimensional effective action is [50]

$$
S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \left\{ R_5 - \frac{40}{3}(\partial f)^2 - 20(\partial w)^2 - \frac{1}{2}(\partial \phi)^2 - \mathcal{P} \right\},
\tag{4.1}
$$

where we defined

$$
\mathcal{P} = -24 e^{-\frac{16}{3}f - 2w} + 4 e^{-\frac{16}{3}f - 12w} + \frac{1}{2} K^2 e^{-\frac{40}{3}f}.
\tag{4.2}
$$

We set the asymptotic AdS radius to one, which corresponds to setting

$$
K = 4.
\tag{4.3}
$$

From Eq. (4.1) we obtain the following equations of motion

$$
0 = \Box f - \frac{3}{80} \frac{\partial \mathcal{P}}{\partial f},
\tag{4.4}
$$

$$
0 = \Box w - \frac{1}{40} \frac{\partial \mathcal{P}}{\partial w},
\tag{4.5}
$$

$$
0 = \Box \phi - \frac{\partial \mathcal{P}}{\partial \phi},
\tag{4.6}
$$

$$
R_{5\mu\nu} = \frac{40}{3} \partial_\mu f \partial_\nu f + 20 \partial_\mu w \partial_\nu w + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{3} g_{\mu\nu} \mathcal{P}.
\tag{4.7}
$$

The uplifted ten dimensional metric takes form

$$
ds_{10}^2 = g_{\mu\nu}(y) dy^\mu dy^\nu + \Omega_1^2(y) e_\psi^2 + \Omega_2^2(y) \sum_{a=1}^{2} \left(e_{\theta_a}^2 + e_{\phi_a}^2 \right),
\tag{4.8}
$$

where y denotes the coordinates of $\mathcal{M}_5$ (Greek indices μ, ν will run from 0 to 4) and the one-forms e_ψ, e_{θ_a}, e_{ϕ_a} $(a = 1, 2)$ are given by

$$e_\psi = \frac{1}{3}\left(d\psi + \sum_{a=1}^{2}\cos\theta_a \ d\phi_a\right), \qquad e_{\theta_a} = \frac{1}{\sqrt{6}}d\theta_a\,, \qquad e_{\phi_a} = \frac{1}{\sqrt{6}}\sin\theta_a \ d\phi_a\,. \quad (4.9)$$

Furthermore,

$$g_{\mu\nu}(y)dy^\mu dy^\nu = \Omega_1^{-2/3}\Omega_2^{-8/3} \ ds^2\,, \qquad \Omega_1 = e^{f-4w}\,, \qquad \Omega_2 = e^{f+w}\,, \qquad (4.10)$$

and ds^2 is the five-dimensional metric (3.11).

4.2 Late-time expansion and solution

We consider the same five-dimensional metric ansatz as in (3.11); we use the late-τ expansion of the metric warp factors $\{a, b, c\}$ as in (3.21). In order to avoid pole singularities in curvature invariants, the dilaton must be set to zero. For the supergravity scalar $f(\tau, v)$ dual to a dimension-8 operator in KW plasma and for the supergravity scalar $w(\tau, v)$ dual to a dimension-6 operator in KW plasma we use the asymptotics

$$\begin{aligned}
f(\tau, v) &= \frac{1}{\tau^2}f_3(v) + \mathcal{O}\left(\tau^{-8/3}\right) \\
w(\tau, v) &= \frac{1}{\tau^2}w_3(v) + \mathcal{O}\left(\tau^{-8/3}\right)\,.
\end{aligned} \qquad (4.11)$$

Since the massive supergravity modes $\{f, w\}$ are turned on only at order $\mathcal{O}(\tau^{-2})$, the metric warp factors $\{a_i(v), b_i(v), c_i(v)\}$ are exactly the same as for the $\mathcal{N} = 4$ SYM plasma, see (3.26)-(3.28), (3.29), (3.31) and (3.33). Moreover, analogously to (3.32)

$$f_3(v) = f_{3,0}\left(\left(\frac{1}{96v^4} + \frac{v^4}{864}\right)\ln\frac{3+v^4}{3-v^4} - \frac{1}{144}\right). \qquad (4.12)$$

For w_3 we find the following equation

$$0 = w_3'' + \frac{5v^8 + 27}{v(v^8 - 9)}\,w_3' - \frac{12}{v^2}\,w_3\,. \qquad (4.13)$$

Solution to w_3 must have a vanishing non-normalizable mode as $v \to 0$. Near the horizon (3.30), the most general solution to (4.13) takes form

$$w_3 = w_{0,0} + w_{0,1}\ln\left(4\ 3^{3/4}\ y\right) + \mathcal{O}\left(y\right)\,. \qquad (4.14)$$

We find that vanishing of the non-normalizable mode of w_3 as $v \to 0$ requires

$$w_{0,0} = w_{0,1} \times \left[2\gamma - \ln\left(\frac{3}{2}\right) + \psi\left(\frac{1}{2}\right) + \psi\left(\frac{3}{2}\right) \right], \tag{4.15}$$

in which case,

$$w_3 = -\frac{\pi}{6\sqrt{3}} \times w_{0,1} \, v^6 + \mathcal{O}\left(v^{14}\right). \tag{4.16}$$

4.3 Quadratic curvature invariants of (4.8)

We collect here results for $\mathcal{R}$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}$ curvature invariants of the metric (4.8) to the third order in the late-τ expansion.

4.3.1 At leading order

$$\mathcal{R}\Big|^{(0)} = -20 + 20, \tag{4.17}$$

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\Big|^{(0)} = 80 + 80, \tag{4.18}$$

$$\mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}\Big|^{(0)} = \frac{8(5v^{16} + 60v^{12} + 1566v^8 + 540v^4 + 405)}{(3 + v^4)^4} + 136, \tag{4.19}$$

where in (4.17)-(4.19) we separated the AdS_5 and the $T^{1,1}$ contributions.

4.3.2 At first order

$$\mathcal{R}\Big|^{(1)} = 0, \tag{4.20}$$

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\Big|^{(1)} = 0, \tag{4.21}$$

$$\mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}\Big|^{(1)} = \frac{1}{\tau^{2/3}} \frac{41472(v^4 - 3)v^8}{(3 + v^4)^5} \, \eta_0. \tag{4.22}$$

4.3.3 At second order

$$\mathcal{R}\Big|^{(2)} = 0, \tag{4.23}$$

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\Big|^{(2)} = 0, \tag{4.24}$$

$$\mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}\Big|^{(2)} = \frac{1}{\tau^{2/3}}\left\{ -\frac{576(v^4-3)v^8}{(3+v^4)^5}C \right.$$

$$+ \frac{6912(5v^{24} - 60v^{20} + 2313v^{16} - 6912v^{12} + 26487v^8 - 18468v^4 + 13851)v^8\eta_0^2}{(3-v^4)^4(3+v^4)^6} \quad (4.25)$$

$$\left. - \frac{4608(5v^{16} + 6v^{12} + 162v^8 + 54v^4 + 405)v^{10}}{(3-v^4)^4(3+v^4)^5} \right\}.$$

4.3.4 At third order

Provided that C is chosen as in (3.33), we find that

$$\mathcal{R}\Big|^{(3)} = 0 \,, \quad (4.26)$$

once we use Einstein equations. Also, it is easy to determine that

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\Big|^{(3)} = -f_{3,0} \times \frac{100}{27}\left(\left(v^4 + \frac{9}{v^4}\right)\ln\frac{3+v^4}{3-v^4} - 6\right). \quad (4.27)$$

The non-singularity condition therefore requires that

$$f_{30} = 0 \,. \quad (4.28)$$

Finally, using (3.31), (4.14) and (4.28), we find

$$\mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\mu\nu\rho\lambda}\Big|^{(3)} = \left(8\ 2^{1/2}\ 3^{3/4} - 384w_{0,1}\right)\ \ln(3-v^4)$$

$$- \left(\frac{20}{3}\ 2^{1/2}\ 3^{3/4}\ \ln\left(6\ (3e)^{1/5}\right) + 384w_{0,0}\right) + \mathcal{O}\left((3-v^4)\right). \quad (4.29)$$

Thus, choosing

$$w_{0,1} = \frac{2^{1/2}\ 3^{3/4}}{48} \,, \quad (4.30)$$

the logarithmic singularity in (4.29) is removed.

4.4 Higher order curvature invariants of (4.8)

Let us define shorthand notation for the contractions of the Riemann tensor. For each integer n we have

$$\mathcal{R}^{[2^n]}{}_{\mu\nu\rho\lambda} \equiv \mathcal{R}^{[2^{n-1}]}{}_{\mu_1\nu_1\mu\nu} \cdot \mathcal{R}^{[2^{n-1}]\ \mu_1\nu_1}{}_{\rho\lambda} \,, \quad (4.31)$$

where

$$\mathcal{R}^{[0]}{}_{\mu\nu\rho\lambda} \equiv \mathcal{R}_{\mu\nu\rho\lambda} \,. \quad (4.32)$$

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We further define higher curvature invariants $\mathcal{I}^{[2^n]}$, generalizing (3.25):

$$
\begin{aligned}
\mathcal{I}^{[2^n]} &\equiv \mathcal{R}^{[2^{n-1}]}_{\phantom{[2^{n-1}]}\mu\nu\rho\lambda} \mathcal{R}^{[2^{n-1}]\ \mu\nu\rho\lambda} \\
&= \mathcal{I}_0^{[2^n]}(v) + \frac{1}{\tau^{2/3}} \mathcal{I}_1^{[2^n]}(v) + \frac{1}{\tau^{4/3}} \mathcal{I}_2^{[2^n]}(v) + \frac{1}{\tau^2} \mathcal{I}_3^{[2^n]}(v) + \mathcal{O}(\tau^{-8/3}).
\end{aligned}
\tag{4.33}
$$

With a straightforward albeit tedious computation we can extract logarithmic singularities in $\mathcal{I}_3^{[2^n]}$. For the first couple invariants we find:

$$
\begin{aligned}
\mathcal{I}_3^{[2]} &= -384 \left(w_{0,1} - \frac{1}{48} 3^{3/4} \sqrt{2} \right) \ln\left(3 - v^4\right) + \text{finite} \\[4pt]
\mathcal{I}_3^{[4]} &= -3072 \left(w_{0,1} - \frac{5}{72} 3^{3/4} \sqrt{2} \right) \ln\left(3 - v^4\right) + \text{finite} \\[4pt]
\mathcal{I}_3^{[8]} &= -98304 \left(w_{0,1} - \frac{14}{27} 3^{3/4} \sqrt{2} \right) \ln\left(3 - v^4\right) + \text{finite} \\[4pt]
\mathcal{I}_3^{[16]} &= -50331648 \left(w_{0,1} + \frac{512}{3} 3^{3/4} \sqrt{2} \right) \ln\left(3 - v^4\right) + \text{finite} \\[4pt]
\mathcal{I}_3^{[32]} &= -6597069766656 \left(w_{0,1} + \frac{1375731712}{27} 3^{3/4} \sqrt{2} \right) \ln\left(3 - v^4\right) + \text{finite} \\[4pt]
\mathcal{I}_3^{[64]} &= -5666839779443574256 4352 \left(w_{0,1} + \frac{5044031582654955520}{9} 3^{3/4} \sqrt{2} \right) \\
&\qquad \times \ln\left(3 - v^4\right) + \text{finite},
\end{aligned}
\tag{4.34}
$$

as $v \to 3_-^{1/4}$. Interestingly, we find that all lower order invariants, *i.e.*,

$$
\mathcal{I}_i^{[2^n]}, \qquad n = \{1, 2, 3, 4, 5, 6\}, \qquad i = \{0, 1, 2\},
\tag{4.35}
$$

are finite as $v \to 3_-^{1/4}$.

Clearly, given (4.34), logarithmic singularities of the curvature invariant (4.33) can not be canceled within the supergravity approximation.

5 Conclusion

In this paper, following [25, 27, 28] we attempted to construct a string theory dual to strongly coupled conformal expanding plasmas in Bjorken regime [38]. In order to have computational control we truncated the full string theory to supergravity approximation, and focused on the well-established examples of the gauge/string dualities: we considered $\mathcal{N} = 4$ SYM [1] and superconformal Klebanov-Witten [51] gauge theories. We used non-singularity of the dual gravitational backgrounds as a guiding principle to

identify gauge theory operators that would develop a vacuum expectation value during boost-invariant expansion of the plasma. Truncation to a supergravity sector of the string theory (along with parity invariance in the Bjorken frame) severely restricts a set of such operators. In the case of the $\mathcal{N} = 4$ SYM, there are only two such gauge invariant operators, while for the Klebanov-Witten plasma one has an additional operator. We constructed supergravity dual as a late-time asymptotic expansion and demonstrated that the gravitational boundary stress energy tensor expectation value has exactly the same asymptotic late-time expansion as predicted by Müller-Israel-Stewart theory of transient relativistic kinetic theory [47, 48] for the boost invariant expansion. As an impressive success of this approach, one recovers by requiring non-singularity of the background geometry at leading and first three subleading orders[8] [25, 27, 28] the equation of state for the plasma, its shear viscosity and its relaxation time, in agreement with values extracted from the equilibrium higher point correlation functions. Unfortunately, we showed that logarithmic singularity in the background geometry can not be canceled within the supergravity approximation. Moreover, given that the singularities appear to persist in arbitrary high metric curvature invariants, we suspect that relaxing the constraint of parity invariance in the Bjorken regime would not help. Indeed, relaxing parity invariance would allow for only finite number of additional (massive) supergravity modes, which, as an example of Klebanov-Witten plasma demonstrates, would only allow to cancel logarithmic singularities in finite number of additional metric curvature invariants.

   We would like to conclude with several speculations.

■ It is possible that though the full asymptotic late-time expansion of strongly coupled expanding boost invariant plasma is ill-defined within supergravity approximation, the first couple orders can nonetheless we used to extract transport coefficients, and thus can be of value to RHIC (and future LHC) experiments. To this end we note a curiosity that the relaxation time of the $\mathcal{N} = 4$ and the KW plasma came up to be the same. Moreover turning on the scalar fields did not modify the extracted relaxation time. This might point to the universality of the relaxation time, not too dissimilar to the universality of the shear viscosity in gauge theory plasma at (infinitely) strong 't Hooft coupling (see [53] for a conjecture on a bound on relaxation times). Second, the relaxation time computed is the relaxation time for the equilibration of the shear modes. If there is substantial bulk viscosity at RHIC, one would also need an estimate

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[8]At the third subleading order, logarithmic singularities of the metric curvature invariants remain.

for the corresponding relaxation time — this is where non-conformal gauge/string dualities might be useful.

■ It is possible that the singularity observed in the supergravity approximation is genuine, and can not be cured by string theory corrections. A prototypical example of this is the Klebanov-Tseytlin solution [54]. There, one does not expect a resolution of the singularity by string corrections (while preserving the chiral symmetry and supersymmetry) simply because the corresponding (dual) gauge theory phase does not exist. By the same token, the observed singularity in the string theory dual to strongly coupled boost invariant expanding plasma might indicate that such a flow for a *conformal* plasma is physically impossible to realize. For example, a Bjorken flow of a conformal plasma might always become turbulent at late times. In fact, at weak coupling, rapidity breaking instabilities were found [55, 56]. Although this was far from a hydrodynamical regime, the leftover logarithmic singularity appearing at the third order might be a manifestation of such an effect.

We hope to address these issues in future work.

Acknowledgments

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